The Use of Mapping Techniques to Investigate Mathematical Processing and Cognitive Demand in Problem Solving

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This study investigated the relationship between senior secondary students' written responses and the mathematical and cognitive processing used when solving a complex problem. Response maps were used to categorise scripts by global strategy. These categories were stratified into clusters displaying common elements of mathematical structure and cognitive demand. The maps revealed difficulties faced by students in applying known facts and formulae to complex word problems. The overwhelming reason for failure on the task was inability to construct a satisfactory model of the problem . The majority of students were able to identify and record the essential elements of the problem and to recall necessary formulae and procedural skills for a satisfactory solution but they had difficulty establishing crucial links between the data in their representation. This did not necessarily reduce the cognitive demand of the task the student attempted to solve. Cognitive demand appeared to be more related to global strategy chosen.

The ability to solve mathematical problems is a highly desirable outcome of mathematical instruction in high school classrooms. To gauge whether or not students have acquired this ability, it is common to use questions in pen and paper tests which are designed to test "process" rather than mere recollection of "content" or the stereotypical application of "skill". The part of the study (Stillman, 1993) described in this report focused on the mathematical processing that results in a particular solution to a problem solving test item and the underlying cognitive processing that lead to that mathematical processing.

The development of a mapping procedure by Chick, Watson and Collis (1988) to analyse mathematical responses to reveal their underlying structure has proved a useful addition to the toolkit of researchers in problem solving. The response maps were originally devised to illustrate the SOLO levels (Biggs & Collis, 1982) of responses to typical mathematical questions posed in high school classrooms. The usefulness of the maps in examining students' responses and classifying errors using a scheme developed by Chick et al(1988) has been demonstrated with area problems (Watson, Chick & Collis, 1988), a polynomial problem (Chick, 1988) and early multiplication word problems (Watson & Mulligan, 1990). Taplin (1991) has also used them to observe perserverance patterns during problem solving whilst recently Watson (in press) used them to explore strategies for problem solving.

Chick (1988) suggested a further use for the mapping technique would be task analysis maps because of the power of the technique "to visually represent and highlight the structural and procedural steps in a solution" (p.108). Before using a problem as an assessment item, the task analysis map can be used on the model solution to identify the data and the processes that are required for a successful solution. The map highlights the mathematical structure of what the examiner perceives as a well reasoned solution and gives a visual indication of the cognitive demand of the task.

In a similar way, when the resulting student responses to the set task are analyzed, response maps illuminate the mathematical structure (or lack of it) in the student's recorded solution. The maps readily identify those components of the data that were employed or ignored, the mathematical processes applied to this data and the apparent ordering of that processing. The maps can then be used to categorize responses according to perceived commonalities.

To obtain a task analysis map, information from the question statement (including any implicit formulae that are essential to obtain a correct solution) is identified and represented by a black dot, "•". Information which is external to the question but which could be useful is identified and represented by an open dot, "o". The student's combining of this information to reach various subgoals in the solution process is shown by the joining of arcs from the various data and processes to nodes representing the subgoals. The solution progresses from left to right across the map with the final conclusion at the far right. Response maps are similar to task analysis maps except that a further symbol, "x", is introduced to indicate inappropriate information either in the question or external to it that the student has identified or used. Figure 1 shows a typical response map displaying the characteristics mentioned above.

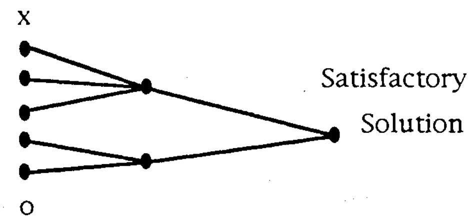

Figure 1. A typical response map.

Chick (1988) includes all the given data elements in the map not distinguishing between the case where not all the data is selected and the case where all the data is selected but not applied. In the study being described, only those elements of the data that have been specifically identified by the student are included in the map as the the distiction is relevant.

Collis and Watson (1991) have recently updated this version of the mapping procedure. However, the more recent version appears to produce diagrams of considerable complexity even with simple word problems so it was decided not to attempt to use these with complex word problems as it was thought that their complexity could promote teacher resistance to their use.

PURPOSE OF THE STUDY

The specific research question reported in this paper was:
Is it possible to identify characteristic response types in written scripts from mathematical problem solving sessions and to classify them in terms of mathematical and cognitive attributes?

METHODS

Two problems were posed in the study but only the Famine problem (see Appendix) will be examined here. The sample consisted of the scripts from 121 Year 11 Mathematics I students at a Catholic Girls' School. The ages of the students ranged from 15 years 7 months to 17 years 10 months. These students were asked to solve the Famine Problem under formal examination conditions and were allowed 25 minutes to complete the task.

Analysis of Task

This problem can be solved by a variety of methods. Figure 2 shows a task analysis map of the particular solution provided by the Mathematics Co-ordinator in her marking scheme. The problem was presented in a purely written format and, as can be seen from the task analysis map, entailed a large complex data set for initial perception, representation and analysis. The problem was expected to tax the problem solver's memory capacity through the size of the data set and the number of variables and relationships stated explicitly and implied. There would be a considerable demand on the student's working memory and it was, therefore, expected that students would augment working memory with an external representation such as pen and paper. It was envisaged that successful solutions would require sustained abstract reasoning supported by efficient memory and resource management.

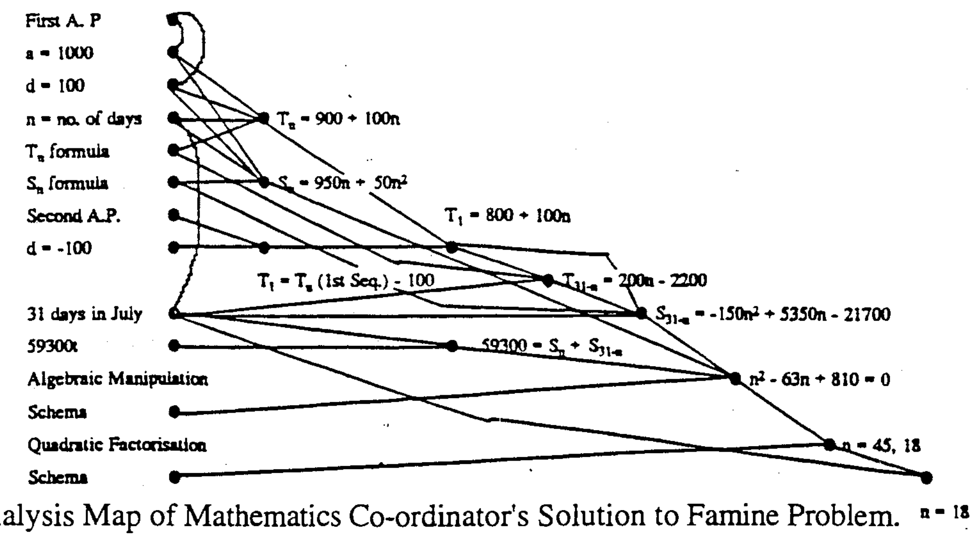

Figure 2. Task Analysis Map of Mathematics Co-ordinator's Solution to Famine Problem.

Strategies Used in Solving the Problem

Analysis of the student scripts revealed that students produced four potentially successful global solution strategies which will be referred to as Strategies A, B, C, and D. The remaining scripts were all pathologically flawed by the global strategy employed and these have been identified as strategies E, F and G. The response clusters produced by using these particular strategies have been categorised accordingly, and then stratified within each category into clusters displaying particular common elements. Table 1 below shows this categorization and the frequency of the various strategies.

Table 1

Categorisation of Examination Scripts for the Famine Problem by Global Strategy

Strategy	Description	Frequency
A	Addition of two Arithmetic Progressions	36
B	Difference method	18
C	Numeric method	4
D	Formalised trial and improvement	5
E	One Arithmetic Progression	52
F	Geometric Progression /Arithmetic and a Geometric Progression	4
G	Averaging	2
	TOTAL	121

As it is well beyond the length restriction of this paper to deal with all the strategies, the analysis of those scripts employing strategy A will be examined as an illustration of the technique used and its utility. In Table 2 scripts using Strategy A are further classified into clusters according to a perceived pattern of mathematical structure as revealed by the response maps.

Table 2

Classification of responses using Strategy A by mathematical structure

Cluster	Description	Frequency
a	Appropriate model with no missing links	1
b(i)	Basis for appropriate model but one misssing link	7
b(ii)	Basis for appropriate model but two missing links	18
b(iii)	Basis for appropriate model but three missing links	4
c(i)	Appropriate global model but use of inappropriate submodel	1
c(ii)	Appropriate global model but inappropriate data & poor algbebraic skills	1
d	Little structure - grasping at straws.	4
	TOTAL	36

Of the 36 scripts using Strategy A, three (8.3%) had no missing links in the student's representation of the problem. Seven (19.4%) had one missing link, 18 (50%) had two whilst 8 (22.2%) had three. The nature of these missing links appeared to contribute to the difficulty of the problem with three types of links proving to be vital for constructing an appropriate global model. By far the most difficult link to make was the relationship between the last term of the first A.P. and the first term of the second A.P. This link was missing in 30 (83.3%) of the scripts using strategy A and was the missing link in 6 of the 7 scripts that had just one missing link. The second most difficult link to establish was the relationship between the number of terms in the two A.P.s with 26 (72.3%) of the scripts having this error. The third most difficult link to establish was the link between the sums of the two A.P.s with 10 (27.8%) of scripts having this error but this particular link was not omitted in isolation.

The order of difficulty for establishing these links was not anticipated by the teachers who had set the task. The marking scheme provided awarded one mark for finding the link between the sums of the two A.P.'s, half a mark for the link between the number of terms in the two A.P.'s whilst no marks were explicitly allocated for the link between the last term of the first A.P. and the first term of the second A.P. This scheme is in the reverse order of difficulty for establishing these links as revealed by the foregoing analysis of the scripts.

A comparison of the response maps of scripts with one (Figures 3 & 4), two (Figure 5) and then three (Figure 6) missing links showed that the cognitive demand of the task the students attempted was not necessarily reduced by the number of links that were missing nor the nature of the missing link. The effects of these missing links on the reasonableness of the results became more marked as two and then three links were omitted. There were, however, other contributing factors such as the use of inappropriate data or procedures and/or difficulty with algebraic skills.

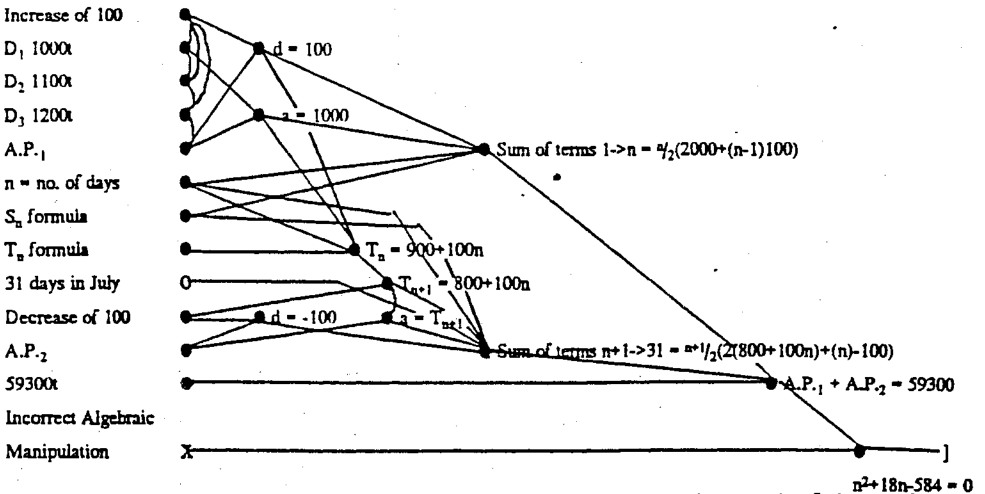

Figure 3. Map of a student's solution showing that the cognitive demand of the task has not been decreased with one missing link in the model between the number of terms of the AP's.

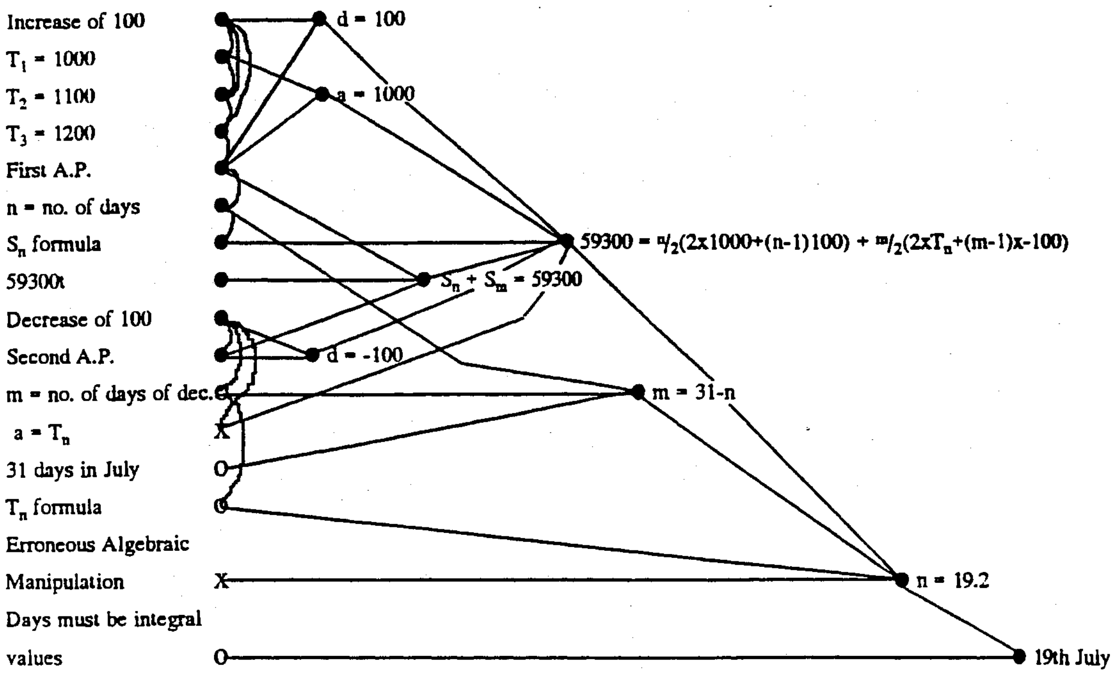

Figure 4. Map of a student's solution showing that the nature of the missing link (between the last term of first AP and first term of second AP) has not reduced the cognitive demand of the task.

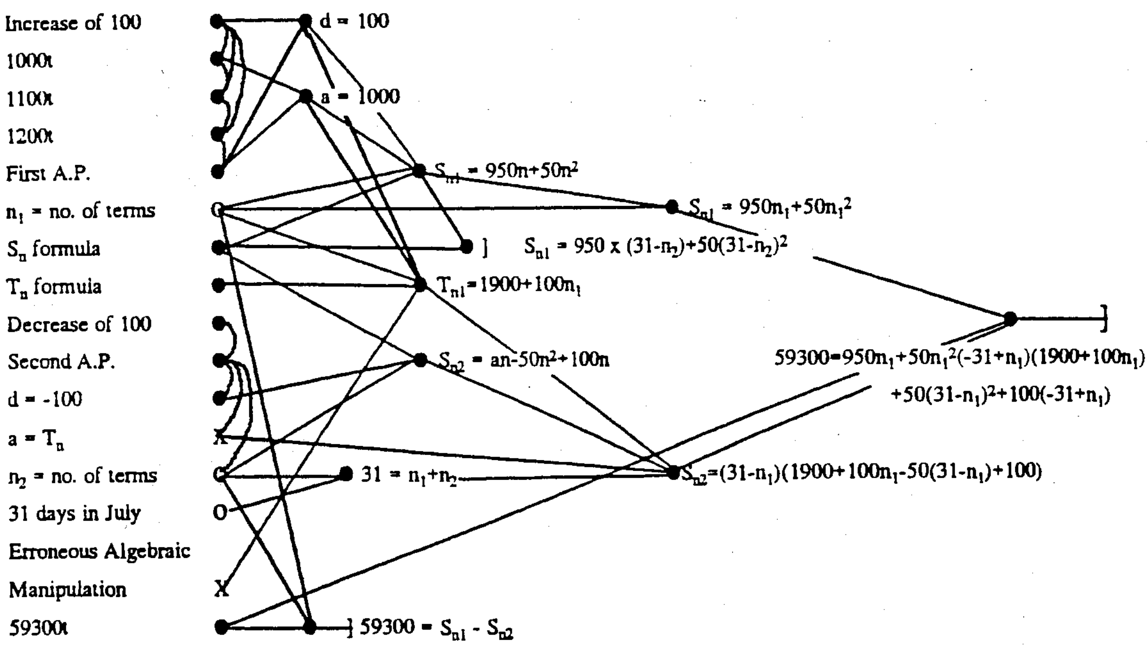

Figure 5. Map of student's solution where two links are missing from the model.

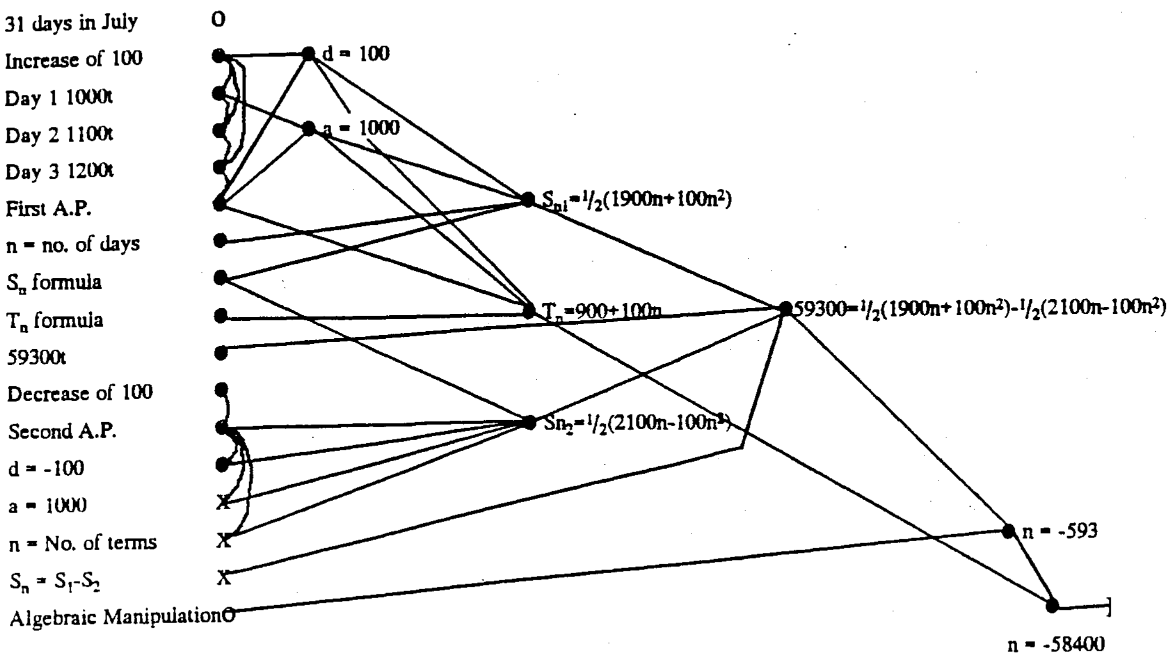

Figure 6. Map of student's solution where three links are missing from the model.

Comparison of the response maps of scripts using different strategies, showed that cognitive demand was more related to the particular global strategy chosen. Strategy E, in particular, which was employed in 52 (43%) of the scripts considerably reduced the mathematical processing and the cognitive demand of the task as can be seen in Figure 7.

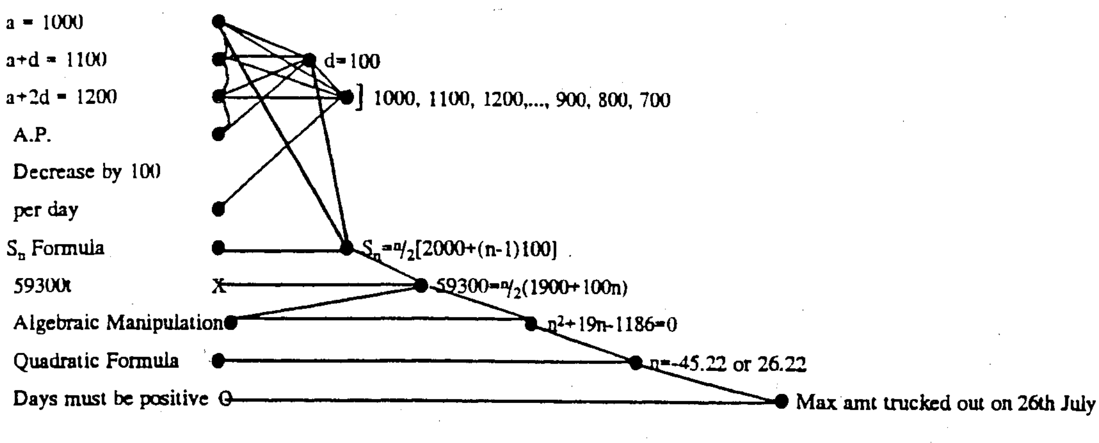

Figure 7. Map of a student's solution using Strategy E.

DISCUSSION OF RESULTS AND IMPLICATIONS

It was possible using the response map technique to identify characteristic response types in the written scripts and to classify them in terms of their mathematical and cognitive attributes. The lack of recorded detail in some of the numeric solutions obviated the use of the maps in that instance. The maps themselves provided a visual summary of the analysis of the scripts allowing comparisons such as cognitive demand of particular strategies. The maps also revealed a large number of the difficulties students face in applying known facts and formulae in word problems of this complexity. By being made aware of the possibility for such difficulties, teachers can be more alert to them should they arise in the classroom and can also draw students' attention to these possible sources of error. Remediation can also be facilitated as the maps reveal and record the precise nature of students' difficulties.

Even though some students based their solution on a small subset of the data, the majority of students in the study were able to select the relevant information from the problem statement and to recall the necessary formulae from LTM but this was no guarantee of success. The less successful students were content to look at the surface characteristics of the problem whereas the more successful problem solvers focused on the underlying structure of the relationships between the data in order to attempt to develop a meaningful representation of the problem. This observation is consistent with the findings of Lester, Garofalo and Kroll (1989) and the expert-novice studies of Chi, Glaser and Rees (1982) and Schoenfeld and Hermann (1982).

The overwhelming reason for failure on the problem was the inability of students to construct a satisfactory model of the problem situation no matter which particular global strategy was chosen. This finding confirms much previous research (eg, Chi, Glaser, and Rees, 1982). The majority of students had great difficulty establishing crucial relationships between the data in their problem representations. Contributing factors to this situation were unsatisfactory comprehension skills, inability to inhibit impulsive responses to the problem, and a lack of understanding of the mathematical concepts involved despite being able to correctly recall formulae. Solutions were further hampered by poor algebraic manipulative skills particularly those associated with the solution of quadratic equations.

Clearly, the findings of this present study have implications for both classroom practice and future research. Much greater attention must be paid to the critical need to construct a satisfactory model of the problem situation. A future research question could be: In what ways do teachers (as evidenced by their classroom practice or their marking schemes) pay attention to the satisfactory construction of problem representations?

REFERENCES

Biggs, J.B., & Collis, K.F. (1982). *Evaluating the Quality of Learning: The SOLO Taxonomy.* New York: Academic Press.

Chi, M., Glaser, R. & Rees, E. (1982). Expertise in problem solving. In R.Sternberg (Ed.), *Advances in the Psychology of Human Intelligence*. Hillsdale, NJ: Erlbaum.

Chick, H. L. (1988). Student responses to a polynomial problem in the light of the SOLO Taxonomy. *Australian Senior Mathematics Journal, 2*(2), 91-110.

Chick, H.L., Watson, J.M., & Collis, K.F. (1988). Using the SOLO Taxonomy for error analysis in mathematics. *Research in Mathematics Education in Australia*, June, 34-46.

Collis, K.F. & Watson, J.M. (1991). A mapping procedure for analysing the structure of mathematics responses. *Journal of Structural Learning, 11*(1), 65-87.

Lester, F.K., Garofalo, J. & Kroll, D.L. (1989). *The Role of Metacognition in Mathematical Problem Solving*. Final Report. Bloomington, IN: Indiana University, Mathematics Education Development Centre. (ERIC Document Reproduction Service No. ED314 255).

Schoenfeld, A.H. & Hermann, D.J. (1982). Problem perception and knowledge structure in expert and novice mathematical problem solvers. *Journal of Experimental Psychology: Learning, Memory and Cognition, 8*(5), 484-494.

Stillman, G.A. (1993). *Metacognition and Mathematical Learning: a Problem Solving Application*. Unpublished master's thesis, University of Queensland, Brisbane.

Taplin, M.L. (1992). *An Investigation of Perseverance in Number Problem Solving.* Unpublished doctoral dissertation, University of Tasmania.

Watson, J.M. (in press). A diagrammatic representation for studying problem solving behaviour. *Journal of Mathematical Behaviour.*

Watson, J.M., Chick, H.L., & Collis, K.F. (1988). Applying the SOLO Taxomony to errors on area problems. In J. Pegg (Ed.), *Mathematical Interfaces* (pp. 260-281). Armidale, N.S.W.: Australian Association of Mathematics Teachers.

Watson, J.M., & Mulligan, J. (1990). Mapping solutions to an early multiplication word problem. *Mathematics Education Research Journal, 2*(2), 28-44.

APPENDIX

The Famine Problem

Every day during the month of July, Relief Aid Abroad trucked supplies of food into the famine stricken areas of Nacirema. On the first day, 1000 tonnes were shifted; on the second day, 1100 tonnes were shifted; on the third day, 1200 tonnes were shifted and so on until a maximum amount was reached. The supply of food then declined by 100 tonnes per day until the end of the month. If the total food supplied for the month was 59300 tonnes, on which day of the month was the maximum amount trucked out?